J. Fluid Mech. (1966), *vol.* 26, *part* 3, *pp.* 433–436
Printed in Great Britain

The vorticity jump across a flow discontinuity

By SUNE B. BERNDT
Royal Institute of Technology, Stockholm

(Received 28 February 1966)

Expressions are obtained for the jump in vorticity across a discontinuity surface which is not a contact surface. Admitting arbitrary motion of the fluid and the discontinuity, the most general of these is established on a purely kinematical basis. More specific expressions are obtained by successively enforcing the momentum equation of the flow and the conditions of conservation of mass and momentum across the discontinuity. Eventually Hayes's (1957) result for a gasdynamic discontinuity is recovered.

1. Introduction

Hayes (1957) has derived an expression for the vorticity jump across gasdynamic discontinuities, such as shock waves, which is of rather general validity and includes as special cases all previous results of the same type. The discontinuity surface, embedded in the unsteady, inviscid flow field of an otherwise completely unspecified fluid, may move in an arbitrary manner, while, across the surface, conservation of energy is *not* implied, only conservation of mass and momentum. The purpose of this paper is to establish, on a purely kinematical basis, a vorticity jump formula of complete generality. By introducing the appropriate dynamical equations and the condition of conservation of mass we obtain from this formula a series of generalizations of Hayes's result. Thus, for example, one can account for impulsive forces acting on the fluid when it crosses the discontinuity.

It is evident from the analysis of Hayes (1957) that the task of computing the vorticity jump across a gasdynamic discontinuity is greatly facilitated by employing a reference frame in which the discontinuity is a normal one. The most general expression which can be obtained in this way extends Hayes's result to include the effect of an extraneous force field and a *normal* impulsive force (Berndt 1966). In the present case, where we must allow for a tangential velocity jump, there is no reference frame of the type required, and hence one would expect considerable complications to arise. It turns out, however, that the simplicity of the previous analysis is retained by the simple stratagem of using simultaneously two different frames of normal flow, one for each side of the discontinuity surface.

2. Kinematical considerations

We shall be concerned with fields, in three-dimensional Euclidean space, which have jump discontinuities across a surface S of continuous curvature and smooth motion, but which are otherwise continuously differentiable outside S, as well as on the two faces of S. In computing jumps across S, at a point P and time t, the analysis will be purely local, so it is sufficient that the foregoing description is valid in a neighbourhood of P and t.

Introducing the unit normal vector $\mathbf{n}$ of S, a continuously differentiable function of distance along S, we shall denote by $[a]$ the jump of any quantity a, in crossing S in the sense of $\mathbf{n}$. The normal and tangential parts of a vector $\mathbf{a}$ at S will be denoted by $\mathbf{a}_n$ or $a_n\mathbf{n}$, and $\mathbf{a}_S$, respectively.

Our strategy when determining the vorticity jump will be to express it in terms of such quantities as appear naturally in the laws of conservation across S, primarily the flow velocity. We shall accept derivatives of such quantities only to the extent that they can be evaluated without leaving S.

The vorticity jump $[\boldsymbol{\zeta}]$ is independent of the motion of any rigid frame with respect to which the vorticity $\boldsymbol{\zeta}$ is defined. As indicated, we shall be using two different rigid frames for the flow at P, one for each side of S. If $\mathbf{u}$ is the flow velocity and curl $\mathbf{u}$ the vorticity with respect to such a local frame, then

$$[\boldsymbol{\zeta}] = [\operatorname{curl}\mathbf{u}]+2[\boldsymbol{\omega}], \tag{1}$$

where $\boldsymbol{\omega}$ is the angular velocity of the local frame with respect to some common frame. We shall need only the tangential part of (1), since the normal part of the vorticity jump can be readily obtained in the proper form. Indeed, the definition of vorticity as a circulation integral implies that

$$[\boldsymbol{\zeta}_n] = \operatorname{Curl}[\mathbf{v}_S], \tag{2}$$

where Curl is the curl operator in S, and $\mathbf{v}$ is the flow velocity with respect to a common frame.

In order to obtain an expression for the tangential part of $[\boldsymbol{\zeta}]$ we shall employ the kinematical vector identity (as did Uberoi, Kuethe & Menkes 1958)

$$D\mathbf{u} = \partial\mathbf{u}/\partial t+\operatorname{grad}\tfrac{1}{2}\mathbf{u}^2-\mathbf{u}\times\operatorname{curl}\mathbf{u}, \tag{3}$$

where D denotes material differentiation. So as to remain in S in evaluating the gradient we can only use the tangential part of (3). The tangential part of the acceleration $D\mathbf{u}$ can be accepted since it is related to the force and pressure distribution *along* S.

For complete simplicity we choose the local frame on either side of S to move in such a way that the tangential velocity $\mathbf{u}_S$ vanishes at a point Q, fixed in the frame: Q, as it moves, remains in S and at time t coincides with P. In addition we assume that the frame turns in such a way that the normal $\mathbf{n}$ at Q remains fixed with respect to it. Except for rotation around $\mathbf{n}$ at Q, this prescription determines uniquely two local frames, one on each side of S; they move relative to each other if there is a jump in $\mathbf{u}_S$, but at time t they effectively coincide.

At Q then, where the contribution of $\mathbf{u}_S$ to the tangential part of grad $\mathbf{u}^2$ vanishes, equation (3) yields the simple expression

$$(D\mathbf{u})_S = u_n(\operatorname{Grad}u_n-\mathbf{n}\times\operatorname{curl}_S\mathbf{u}) \tag{4}$$

for the tangential part of the acceleration. Here Grad is the gradient operator in S, while $\operatorname{curl}_S\mathbf{u}$ is the tangential part of curl $\mathbf{u}$. Taking jumps at P and t, and noting that $[u_n] = [v_n]$, we find that

$$[\operatorname{curl}_S\mathbf{u}] = \mathbf{n}\times\{[(D\mathbf{u})_S/u_n]-\operatorname{Grad}[v_n]\}, \tag{5}$$

provided the normal velocity component u_n is different from zero on *both* sides. Finally, from (1) and (2) we obtain a formula of the type required:

$$[\boldsymbol{\zeta}] = \mathbf{n}\times\{[(D\mathbf{u})_S/u_n] - \operatorname{Grad}[v_n]\} + \operatorname{Curl}[\mathbf{v}_S] + 2[\boldsymbol{\omega}_S] \quad \text{at } P \text{ and } t. \tag{6}$$

This completely general expression for the vorticity jump is of a purely kinematical origin. It owes its formal and conceptual simplicity to our special choice of local reference frames for the acceleration of the flow on either side of S.

3. Dynamical considerations

Turning now to the dynamics of the flow, we first introduce the momentum equation for *inviscid* flow at either side of S. Let $\mathbf{a}$ and $\boldsymbol{\omega}$ be the acceleration of Q and the angular velocity of the local frame with respect to an *inertial* frame, respectively. We then have at Q

$$D\mathbf{u} + \frac{1}{\rho}\operatorname{grad} p = \mathbf{f} - \mathbf{a} + 2u_n\,\mathbf{n}\times\boldsymbol{\omega}, \tag{7}$$

where ρ and p are the density and pressure of the fluid and $\mathbf{f}$ is an extraneous force field (force per unit mass).† Considering tangential parts only, we obtain at P and t

$$[(D\mathbf{u})_S/u_n] = -[(1/\rho u_n)\operatorname{Grad} p] + [(\mathbf{f}_S - \mathbf{a}_S)/u_n] + 2\mathbf{n}\times[\boldsymbol{\omega}_S] \tag{8}$$

to be substituted into (6). Hence it follows that the vorticity jump at P and t is

$$[\boldsymbol{\zeta}] = \mathbf{n}\times\{[(1/\rho)\operatorname{Grad} m] - [(1/m)\operatorname{Grad}(p + mu_n)] + [\rho(\mathbf{f}_S - \mathbf{a}_S)/m]\} + \operatorname{Curl}[\mathbf{v}_S], \tag{9}$$

where, in anticipation of the conservation conditions across S, the flux density of mass,

$$m = \rho u_n, \tag{10}$$

and of normal momentum, mu_n, have been introduced. This expression for $[\boldsymbol{\zeta}]$ is still quite general, however; it does not imply conservation of either mass, momentum or energy across S.

Next assume conservation of mass. Then m must be continuous across S, and (9) simplifies to

$$[\boldsymbol{\zeta}] = \mathbf{n}\times\{[1/\rho]\operatorname{Grad} m + [\rho(\mathbf{f}_S - \mathbf{a}_S)]/m\} - (1/m)\,\mathbf{n}\times\operatorname{Grad} F_n + \operatorname{Curl}(\mathbf{F}_S/m) \quad \text{at } P \text{ and } t, \tag{11}$$

where

$$\mathbf{F} = [p + mu_n]\,\mathbf{n} + m[\mathbf{v}_S] \quad \text{on } S \text{ at } t. \tag{12}$$

In considering the balance of the flow of momentum across S, with inviscid flow on either side, we realize that $\mathbf{F}$ should be interpreted as an impulse per unit area and time received by the fluid in crossing S (if this occurs in the sense of $\mathbf{n}$). This, then, is the vorticity-jump formula relevant to the flow across an 'actuator disk', a wire gauze, or an electromagnetic shock wave.

For ordinary gasdynamic discontinuities we have $\mathbf{F} = 0$. The local frames are then identical, and the vorticity jump simplifies to (Berndt 1966)

$$[\boldsymbol{\zeta}] = \mathbf{n}\times\{[1/\rho]\operatorname{Grad} m + (\mathbf{f}_S - \mathbf{a}_S)[\rho]/m\} \quad \text{at } P \text{ and } t, \tag{13}$$

† For full generality $\rho\mathbf{f}$ might be taken to include the divergence of a viscous stress tensor.

if $\mathbf{f}$ is continuous. For $\mathbf{f} = 0$ this is essentially the formula obtained by Hayes (1957), as is seen by expressing $\mathbf{a}_S$ in terms of the motion of the fluid and the discontinuity surface with respect to an inertial frame.

We have not assumed conservation of energy across S (so the results are still valid, should there be an impulsive heat addition to the fluid when it crosses S). However, by employing the conservation of energy, we can eliminate the pressure from (9) or (11) without having to consider the *normal* momentum balance. This is done by introducing the thermodynamic law connecting the pressure gradient with the gradients of enthalpy and entropy, which means that we bring the thermodynamic properties of the fluid into the picture. The resulting expression for the vorticity jump (Berndt 1966) generalizes earlier results obtained from Crocco's vorticity law.

REFERENCES

BERNDT, S. B. 1966 The vorticity jump across a gasdynamic discontinuity as influenced by extraneous forces. *Recent Progress in Applied Mechanics* (ed. B. Broberg *et al.*). Stockholm: Almqvist and Wiksell/Gebers.

HAYES, W. D. 1957 The vorticity jump across a gasdynamic discontinuity. *J. Fluid Mech.* **2**, 595.

UBEROI, M. S., KUETHE, A. M. & MENKES, H. R. 1958 Flow field of a Bunsen flame. *Phys. Fluid.* **1**, 150.